

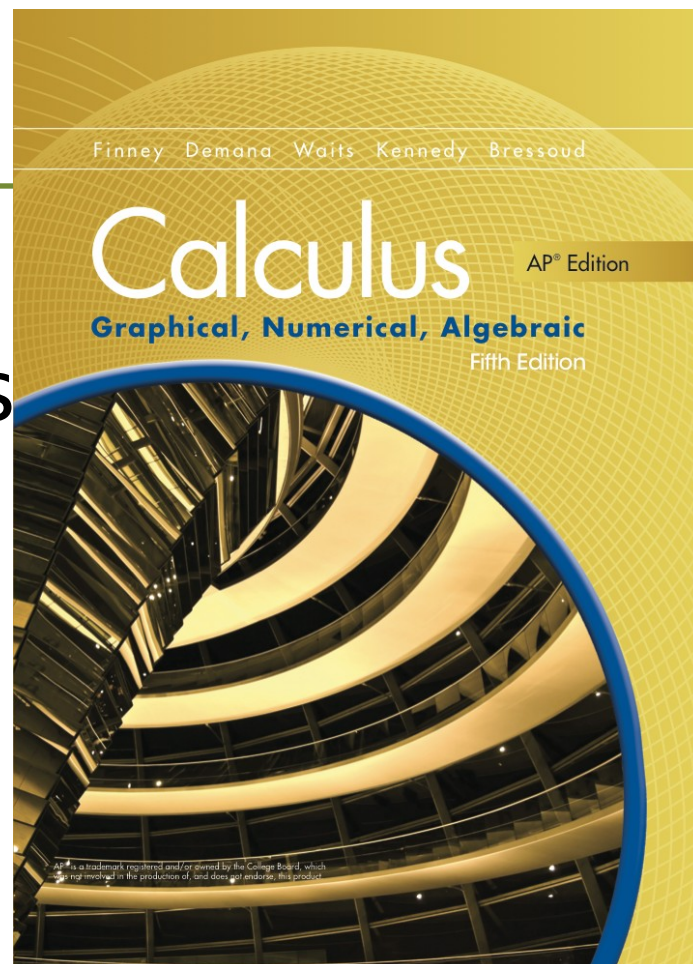
Chapter 4

More Derivatives



Section 4.3

Derivatives of Inverse Trigonometric Functions



Quick Review

In Exercises 1–5, give the *domain* and *range* of the function, and evaluate the function at $x=1$.

1. $y=\sin^{-1} x$

2. $y=\cos^{-1} x$

3. $y=\tan^{-1} x$

4. $y=\sec^{-1} x$

5. $y=\tan(\tan^{-1} x)$

Quick Review

In Exercises 6–10, find the inverse of the given function.

6. $y = 3x - 8$

7. $y = \sqrt[3]{x + 5}$

8. $y = \frac{8}{x}$

9. $y = \frac{3x - 2}{x}$

10. $y = \arctan\left(\frac{x}{3}\right)$

Quick Review Solutions

In Exercises 1– 5, give the *domain* and *range* of the function, and evaluate the function at $x=1$.

1. $y=\sin^{-1} x$ Domain: $[-1, 1]$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ At 1: $\frac{\pi}{2}$
2. $y=\cos^{-1} x$ Domain: $[-1, 1]$ Range: $[0, \pi]$ At 1: 0
3. $y=\tan^{-1} x$ Domain: All Reals Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ At 1: $\frac{\pi}{4}$
4. $y=\sec^{-1} x$ Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ At 1: 0
5. $y=\tan(\tan^{-1} x)$ Domain: All Reals Range: All Reals At 1: 1

Quick Review Solutions

In Exercises 6–10, find the inverse of the given function.

$$6. \quad y = 3x - 8 \qquad f^{-1}(x) = \frac{x+8}{3}$$

$$7. \quad y = \sqrt[3]{x+5} \qquad f^{-1}(x) = x^3 - 5$$

$$8. \quad y = \frac{8}{x} \qquad f^{-1}(x) = \frac{8}{x}$$

$$9. \quad y = \frac{3x-2}{x} \qquad f^{-1}(x) = \frac{2}{3-x}$$

$$10. \quad y = \arctan\left(\frac{x}{3}\right) \qquad f^{-1}(x) = 3\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

What you'll learn about

- Differentiating inverse functions
- Graphical interpretation of how the derivative of f is related to the derivative of f^{-1}
- Derivatives of the inverse trigonometric functions
- Identities for graphing arccot, arcsec, and arccsc functions on a graphing calculator

... and why

The relationship between the graph of a function and its inverse allows us to see the relationship between their derivatives.

Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and $f'(x)$ is never zero on I , then f has an inverse and f^{-1} is differentiable at every point on the interval $f(I)$.

If $f^{-1}(a) = b$, the **inverse function slope relationship** relates the derivatives by the equation $(f^{-1})'(a) = \frac{1}{f'(b)}$.

Derivative of the Arcsine

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1.$$

Example Derivative of the Arcsine

If $y = \sin^{-1} 8x^2$, find $\frac{dy}{dx}$.

$$y = \sin^{-1} 8x^2 \quad \text{where } u = 8x^2$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (8x^2)^2}} \left(\frac{du}{dx} (8x^2) \right)$$

$$= \frac{1}{\sqrt{1 - 64x^4}} (16x)$$

$$= \frac{16x}{\sqrt{1 - 64x^4}}$$

Derivative of the Arctangent

The derivative is defined for all real numbers.

If u is a differentiable function of x , we apply the Chain Rule to get

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}.$$

Derivative of the Arcsecant

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx} \sec^{-1} u = \frac{1}{|u| \sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1.$$

Example Derivative of the Arcsecant

Given $y = \sec^{-1}(3x - 4)$, find $\frac{dy}{dx}$.

$$y = \sec^{-1}(3x - 4) \quad \text{where } u = 3x - 4$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{|3x - 4| \sqrt{(3x - 4)^2 - 1}} \frac{d}{dx}(3x - 4) \\ &= \frac{3}{|3x - 4| \sqrt{9x^2 - 24x + 15}} \end{aligned}$$

Inverse Function – Inverse Cofunction Identities

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1} \left(\frac{1}{x} \right)$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\csc^{-1} x = \sin^{-1} \left(\frac{1}{x} \right)$$

Example Derivative of the Arccotangent

Find the derivative of $y = \cot^{-1} x^2$.

Rewrite $y = \cot^{-1} x^2$ as $y = \frac{\pi}{2} - \tan^{-1} x^2$ where $u = x^2$.

$$\frac{dy}{dx} = 0 - \frac{1}{1 + (x^2)^2} \frac{du}{dx} (x^2)$$

$$= - \frac{1}{1 + x^4} (2x)$$

$$= - \frac{2x}{1 + x^4}$$